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Exam I, MTH 221, Spring 2015

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QUESTION 1. Find a matrix A, 2×3, such that

$$\begin{bmatrix}-2 & -1\\ 0 & -2\end{bmatrix} A + 3A = \begin{bmatrix}1 & 1 & 1\\ 0 & 1 & 0\end{bmatrix}$$

$$\left(\begin{bmatrix}-2 & -1\\ 0 & -2\end{bmatrix} + 3\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right) A_{(2\times3)} = \begin{bmatrix}1 & 1 & 1\\ 0 & 1 & 0\end{bmatrix} \Rightarrow \left(\begin{bmatrix}-2 & -1\\ 0 & -2\end{bmatrix} + \begin{bmatrix}3 & 0\\ 0 & 3\end{bmatrix}\right) A = \begin{bmatrix}1 & 1 & 1\\ 0 & 1 & 0\end{bmatrix}$$

$$\Rightarrow \underset{(2\times2)}{\begin{bmatrix}1 & -1\\ 0 & 1\end{bmatrix}} A = \underset{(2\times3)}{\begin{bmatrix}1 & 1 & 1\\ 0 & 1 & 0\end{bmatrix}} \Rightarrow \underbrace{\begin{bmatrix}1 & -1\\ 0 & 1\end{bmatrix}^{-1}\begin{bmatrix}1 & -1\\ 0 & 1\end{bmatrix}}_{I_2} A = \begin{bmatrix}1 & -1\\ 0 & 1\end{bmatrix}^{-1}\begin{bmatrix}1 & 1 & 1\\ 0 & 1 & 0\end{bmatrix}$$

$$\begin{bmatrix}1 & -1\\ 0 & 1\end{bmatrix}^{-1} = \frac{1}{1}\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix} = \begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix} \Rightarrow A = \begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}\begin{bmatrix}1 & 1 & 1\\ 0 & 1 & 0\end{bmatrix} \boxed{= \begin{bmatrix}1 & 2 & 1\\ 0 & 1 & 0\end{bmatrix} = A}$$

QUESTION 2. Given that A is a 5×5 matrix such that $det(A) = 2015$. Let B be the second column of A. Consider the system of L.E $AX = B$. Convince me that such system must have unique solution. Then find the solution.

$$\begin{array}{c|c} & x_1 \quad x_2 \quad x_3 \quad x_4 \quad x_5 \\ \begin{matrix}1\\2\\3\\4\\5\end{matrix} & \begin{bmatrix} & b_1 & & & \\ & b_2 & & & \\ & b_3 & 0 & & \\ & b_4 & & 0 & \\ & b_5 & & & 0\end{bmatrix}\end{array} \begin{bmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{bmatrix} = \begin{bmatrix}b_1\\b_2\\b_3\\b_4\\b_5\end{bmatrix}$$

Since det(A) ≠ 0 then there is an inverse to A and its consistent. Since # of variables is equal to # of equations (m = n) then there are no free variables and the system does not have infinitely many solutions ⇒ it must have a unique solution.

$$\Rightarrow x_1(y_1) + x_2(b_1) + x_3(y_3) + x_4(y_4) + x_5(y_5) = b_1$$

Since the det(A) ≠ 0 ⇒ there are no identical columns, means only the 2nd column of A can be B, therefore the only value x_2 can have is 1 and x_1, x_3, x_4, x_5 have to be zeros.

$$\boxed{\text{Solution set} = \{(0, 1, 0, 0, 0)\}}$$

QUESTION 3. Let $A = \begin{array}{c|c} & 1 \quad 2 \quad 3 \\ \begin{matrix}1\\2\\3\end{matrix} & \begin{bmatrix}2 & 1 & 1\\ -1 & 7 & b\\ 2 & 3 & c\end{bmatrix}\end{array}$ such that $det(A) = -130$. Let $B = \begin{array}{c|c} & 1 \quad 2 \quad 3 \\ \begin{matrix}1\\2\\3\end{matrix} & \begin{bmatrix}2 & 1 & 4\\ -1 & 7 & (b+2)\\ 2 & 3 & c\end{bmatrix}\end{array}$. Find $det(B)$. [Hint use the definition to find det(A) and det(B) and in both calculations use the third column. Then stare.. you might notice something].

$$(1)(-1)^4\begin{vmatrix}-1 & 7\\ 2 & 3\end{vmatrix} + (b)(-1)^5\begin{vmatrix}2 & 1\\ 2 & 3\end{vmatrix} + (c)(-1)^6\begin{vmatrix}2 & 1\\ -1 & 7\end{vmatrix} \Rightarrow (-3-14) - b(6-2) + c(14+1) = -130$$

$$\Rightarrow -17 - 4b + 15c = -130 \Rightarrow 15c = -130 + 4b + 17$$
$$15c = 4b - 113$$

$$\det(B) = (4)(-1)^4\begin{vmatrix}-1 & 7\\ 2 & 3\end{vmatrix} + (b+2)(-1)^5\begin{vmatrix}2 & 1\\ 2 & 3\end{vmatrix} + (c)(-1)^6\begin{vmatrix}2 & 1\\ -1 & 7\end{vmatrix} \Rightarrow 4(-3-14) - (b+2)(6-2) + c(14+1) =$$

$$-68 - (4b+8) + 15c = \det(B)$$
$$\Rightarrow -4b + 15c - 76 = \det(B)$$
$$\Rightarrow -4b + 4b - 113 - 76 = \det(B)$$
$$-189 = \det(B)$$

$$\Rightarrow \boxed{\det(B) = -189}$$

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QUESTION 4. Given A is a 4×4 matrix such that: $A \xrightarrow{3R_1+R_2\to R_2} B \underset{L_1}{\xrightarrow{4R_2}} C \underset{L_2}{\xrightarrow{R_2\leftrightarrow R_3}} D = \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}$ I_4

$\det(D) = 1$

$\det(C) = -1$

$\det(B) = \frac{1}{4}(-1) = -\frac{1}{4}$

$\det(A) = -\frac{1}{4}$

(i) Find elementary matrices F_1, F_2, F_3 such that $F_1F_2F_3A = D$.

$F_3 = \begin{bmatrix}1&0&0&0\\3&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}$ $F_2 = \begin{bmatrix}1&0&0&0\\0&4&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}$ $F_1 = \begin{bmatrix}1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&1\end{bmatrix}$

(ii) Find elementary matrices L_1, L_2 such that $L_1L_2D = B$.

$L_2 = \begin{bmatrix}1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&1\end{bmatrix}$ $L_1 = \begin{bmatrix}1&0&0&0\\0&1/4&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}$

$A = \begin{bmatrix}1&0&0&0\\-3&0&1/4&0\\0&1&0&0\\0&0&0&1\end{bmatrix}$

(iii) Does A^{-1} exist? if yes find A^{-1}.

yes $\det(A^{-1}) = \dfrac{1}{\det(A)} = \dfrac{1}{(-1/4)} = -4$

$\Rightarrow \left[\begin{array}{cccc|cccc}1&0&0&0&1&0&0&0\\-3&0&1/4&0&0&1&0&0\\0&1&0&0&0&0&1&0\\0&0&0&1&0&0&0&1\end{array}\right] \xrightarrow{3R_1+R_2\to R_2} \left[\begin{array}{cccc|cccc}1&0&0&0&1&0&0&0\\0&0&1/4&0&3&1&0&0\\0&1&0&0&0&0&1&0\\0&0&0&1&0&0&0&1\end{array}\right] \xrightarrow{4R_2} \left[\begin{array}{cccc|cccc}1&0&0&0&1&0&0&0\\0&0&1&0&12&4&0&0\\0&1&0&0&0&0&1&0\\0&0&0&1&0&0&0&1\end{array}\right]$

$\Rightarrow \left[\begin{array}{cccc|cccc}1&0&0&0&1&0&0&0\\0&1&0&0&0&0&1&0\\0&0&1&0&12&4&0&0\\0&0&0&1&0&0&0&1\end{array}\right] \Rightarrow \boxed{\begin{bmatrix}1&0&0&0\\0&0&1&0\\12&4&0&0\\0&0&0&1\end{bmatrix} \text{ is } A^{-1}}$

(iv) Find $det(2A)$ and $det(C^T)$

$(2)^4 \det(A)$

$= 16 \times -\frac{1}{4} \boxed{= -4}$

$\det(C^T) = \det(C)$

$\boxed{= -1}$

(v) If $(A^T)^{-1}$ exist, then find it.

$(A^T)^{-1} = (A^{-1})^T = \begin{bmatrix}1&0&12&0\\0&0&4&0\\0&1&0&0\\0&0&0&1\end{bmatrix}$

(vi) Find the solution set for the system of L.E. $AX = \begin{bmatrix}1\\-1\\2\\4\end{bmatrix}$.

$\begin{bmatrix}1&0&0&0\\-3&0&1/4&0\\0&1&0&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} = \begin{bmatrix}1\\-1\\2\\4\end{bmatrix} \Rightarrow x_1\begin{bmatrix}1\\-3\\0\\0\end{bmatrix} + x_2\begin{bmatrix}0\\0\\1\\0\end{bmatrix} + x_3\begin{bmatrix}0\\1/4\\0\\0\end{bmatrix} + x_4\begin{bmatrix}0\\0\\0\\1\end{bmatrix} = \begin{bmatrix}1\\-1\\2\\4\end{bmatrix}$

$\Rightarrow x_1 = 1$

$-3x_1 + \frac{1}{4}x_3 = -1 \Rightarrow -3 + \frac{x_3}{4} = -1 \Rightarrow \frac{x_3}{4} = 2 \Rightarrow x_3 = 8$

$x_2 = 2$

$x_4 = 4$

$\Rightarrow \boxed{\text{solution set} = \{(1,2,8,4)\}}$

QUESTION 5. Use row operations only in order to find $\begin{bmatrix}1&3&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}2&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&-3&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&3\\-2&1\\4&1\end{bmatrix} \xrightarrow{-3R_2}$

$3R_2+R_1 \to R_1$ $\quad 2R_1 \quad -3R_2$

$$\begin{bmatrix}1&3\\6&-3\\4&1\end{bmatrix} \xrightarrow[2R_1]{} \begin{bmatrix}2&6\\6&-3\\4&1\end{bmatrix} \xrightarrow[3R_2+R_1\to R_1]{} \begin{bmatrix}20&-3\\6&-3\\4&1\end{bmatrix}$$

$$\boxed{= \begin{bmatrix}20&-3\\6&-3\\4&1\end{bmatrix}}$$

QUESTION 6. Find the solution set for

$$x_1 + x_2 + x_3 - x_4 = 6$$
$$2x_1 + 2x_2 + 3x_3 + x_4 = 4$$
$$-x_1 - x_3 + 2x_4 = 8$$

$$\begin{array}{c|c} & \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & B\end{array} \\ \begin{array}{c} R_1\\ R_2\\ R_3\end{array} & \left[\begin{array}{cccc|c}1&1&1&-1&6\\2&2&3&1&4\\-1&0&-1&2&8\end{array}\right]\end{array} \xrightarrow[R_1+R_3\to R_3]{-2R_1+R_2\to R_2} \left[\begin{array}{cccc|c}1&1&1&-1&6\\0&0&1&3&-8\\0&1&0&1&14\end{array}\right] \xrightarrow{-R_2+R_1\to R_1} \left[\begin{array}{cccc|c}1&1&0&-4&14\\0&0&1&3&-8\\0&1&0&1&14\end{array}\right]$$

$$R_3+R_1 \to R_1 \quad \left[\begin{array}{cccc|c}1&0&0&-5&0\\0&0&1&3&-8\\0&1&0&1&14\end{array}\right] \Rightarrow \begin{aligned} x_1 - 5x_4 &= 0\\ x_3 + 3x_4 &= -8\\ x_2 + x_4 &= 14\end{aligned}$$

$\Rightarrow$ 3 leading variables (x_1, x_2, x_3)
and 1 free variable x_4
$\Rightarrow$ infinitely many solutions.

$x_1 = 5x_4$
$x_3 = -8 - 3x_4$
$x_2 = 14 - x_4$

$\Rightarrow$ $\boxed{\text{Solution set} = \{(5x_4, 14-x_4, -8-3x_4, x_4) \mid x_4 \in \mathbb{R}\}}$

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QUESTION 7. For what values of k will the following system be consistent?

$$x_1 + x_2 + x_3 - x_4 = 6$$
$$2x_1 + 2x_2 + kx_3 - 2x_4 = 4$$
$$-x_1 - x_2 + 3x_3 + x_4 = 10$$

$$\begin{array}{c|c} & \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & B \end{array} \\ \begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array} & \left[\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 6 \\ 2 & 2 & k & -2 & 4 \\ -1 & -1 & 3 & 1 & 10 \end{array}\right] \end{array} \Rightarrow \begin{array}{c} -2R_1 + R_2 \to R_2 \\ R_1 + R_3 \to R_3 \end{array} \left[\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 6 \\ 0 & 0 & (k-2) & 0 & -8 \\ 0 & 0 & 4 & 0 & 16 \end{array}\right]$$

$\Rightarrow x_1 + x_2 + x_3 - x_4 = 6$

$(k-2)x_3 = -8 \Rightarrow$

$4x_3 = 16 \Rightarrow x_3 = 4$

$$\tfrac{1}{4}R_3 \left[\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 6 \\ 0 & 0 & (k-2) & 0 & -8 \\ 0 & 0 & 1 & 0 & 4 \end{array}\right]$$

$\Rightarrow (k-2)(4) = -8 \Rightarrow$

$k - 2 = -2 \Rightarrow$

k should be 0

$\boxed{k = 0}$

QUESTION 8. For what values of a, b, c will the following system have unique solution?

$$x_1 + ax_2 + bx_3 = 12$$
$$-2x_1 + 8x_2 + cx_3 = 20$$
$$3x_1 + 3ax_2 + 6x_3 = -6$$

$$\begin{array}{c|c} & \begin{array}{cccc} x_1 & x_2 & x_3 & B \end{array} \\ \begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array} & \left[\begin{array}{ccc|c} 1 & a & b & 12 \\ -2 & 8 & c & 20 \\ 3 & 3a & 6 & -6 \end{array}\right] \end{array} \begin{array}{c} 2R_1 + R_2 \to R_2 \\ -3R_1 + R_3 \to R_3 \end{array} \left[\begin{array}{ccc|c} 1 & a & b & 12 \\ 0 & (2a+8) & (2b+c) & 44 \\ 0 & 0 & (-3b+6) & -42 \end{array}\right]$$

$\Rightarrow x_1 + ax_2 + bx_3 = 12$

$(2a+8)x_2 + (2b+c)x_3 = 44$

$(-3b+6)x_3 = -42$

unique solution $\Rightarrow$ all variables are leading $\Rightarrow 2a + 8 \neq 0 \Rightarrow a \neq -4$

$-3b + 6 \neq 0 \Rightarrow b \neq 2$

$c \in \mathbb{R}$

$\boxed{a \neq -4,\ b \neq 2,\ c \in \mathbb{R}}$

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